

Bounds for the availabilities for multistate monotone systems based on decomposition into stochastically independent modules

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August 26, 2010

Abstract

Multistate monotone systems are used to describe technological or biological systems when the system itself and its components can perform at different operationally meaningful levels. This generalizes the binary monotone systems used in standard reliability theory. In this paper we consider the availabilities and unavailabilities of the system in an interval, i.e. the probabilities that the system performs above or below the different levels throughout the whole interval. In complex systems it is often impossible to calculate these availabilities and unavailabilities exactly, but it is possible to construct lower and upper bounds based on the minimal path and cut vectors to the different levels. In this paper we consider systems which allow a modular decomposition. We analyse in depth the relationship between the minimal path and cut vectors for the system, the modules and the organizing structure. We analyse the extent to which the availability bounds are improved by taking advantage of the modular decomposition. This problem was treated also in Butler (1982) and Funnemark and Natvig (1985), but the treatment was based on an inadequate analysis of the relationship between the different minimal path and cut vectors involved, and as a result was somewhat inaccurate. We also extend to an interval bounds that have previously only been given for availabilities at a fixed point of time.

Key words: Availability bounds, minimal cut vectors, minimal path vectors, modular decomposition, multistate monotone systems

1 Introduction

A multistate monotone system (MMS) (C, ϕ) consists of a set $C = \{1, 2, \dots, n\}$ of components and a structure function ϕ , taking values in the set $S = \{0, 1, 2, \dots, M\}$, where n and M are arbitrary natural numbers. The state of component i belongs to a subset S_i of S , assumed to contain 0 and M , and the state at time t is denoted by $X_i(t)$. The system state is supposed to be a non-decreasing function of the component states, and is given by $\phi(\mathbf{X}(t))$, where by definition $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$. We assume $\phi(0, \dots, 0) = 0$ and $\phi(M, \dots, M) = M$. In accordance with tradition in the field, we consider time points t in some subset $\tau(I)$ of an interval I of interest, with $\tau(I)$ finite and $\tau(I) = I$ being typical special cases. The concept of an MMS generalizes the concept of a binary monotone system (BMS) as treated in Barlow and Proschan (1975). It allows a more refined description of a system than the concept of a BMS, which is often necessary in order to handle complex systems that can perform at different levels. The elements of S and S_i are thought of as representing an ordering of meaningful performance levels. In specific applications it may be more natural to let S and S_i consist of arbitrary real numbers that are directly interpretable as some kind of measurable quantities, but we will not use this kind of state spaces in this paper.

The component performance processes $\{X_i(t), t \in I\}$, are random, possibly stochastically dependent processes involving repair at fixed or random points of time. A full probabilistic analysis of a multistate monotone system over an interval I requires the specification of a full dynamic model of the joint component process $\{\mathbf{X}(t), t \in I\}$. A framework for the specification of such a parametric model is given in Gåsemyr and Natvig (2005). In all but very simple cases analytic calculations are intractable. Gåsemyr and Natvig (2005) outlines a procedure for simulating the process $\{\mathbf{X}(t), t \geq 0\}$, and also a data augmentation procedure for using such simulations in Bayesian estimation of the parameters of the model. A program for simulation of a binary system with independent component processes is presented in Huseby et al. (2010), while a similar program for simulation of a multistate system with independent components is under development, see Huseby and Natvig

(2010).

In complex systems, the above mentioned simulation based probabilistic analysis of the system may be prohibitively costly computationally. In many cases there is also insufficient information to model the dynamic behaviour of the marginal component processes, and even more so the joint process of dependent components. The analysis then has to be based on less accurate information of the system. In this paper we will assume that the component availabilities respectively unavailabilities to level j

$$\begin{aligned} p_{X_i}^j &= P(X_i(t) \geq j \text{ for all } t \in \tau(I)), \\ q_{X_i}^j &= P(X_i(t) < j \text{ for all } t \in \tau(I)), i = 1, \dots, n, j = 1, \dots, M \end{aligned} \quad (1)$$

are known. The corresponding system availabilities and unavailabilities

$$\begin{aligned} p_\phi^j &= P(\phi(\mathbf{X}(t)) \geq j \text{ for all } t \in \tau(I)) \\ q_\phi^j &= P(\phi(\mathbf{X}(t)) < j \text{ for all } t \in \tau(I)), j = 1, \dots, M \end{aligned} \quad (2)$$

can then not be calculated, even in the case of independent components, and we have to resort to upper and lower bounds. In this paper we will focus on lower bounds for the system availabilities at different levels. Lower bounds for unavailabilities are completely analogous. For the binary case, such bounds are studied in Bodin (1970), Esary and Proschan (1970), Barlow and Proschan (1975) and Natvig (1980). The multistate case is considered in Butler (1982), Funnemark and Natvig (1985), Natvig (1986) and Natvig (1993). A comprehensive treatment of the area, based also on the results of the present paper, is given in the forthcoming book Natvig (2011).

The basic bounds given in these publications are based on the sets of minimal path vectors and minimal cut vectors to level j , i.e. vectors \mathbf{y} respectively \mathbf{z} that are minimal respectively maximal in the natural ordering on $S_1 \times \dots \times S_n$ with respect to the properties that $\phi(\mathbf{y}) \geq j$ respectively $\phi(\mathbf{z}) < j$. Such a vector \mathbf{z} is called a minimal rather than a maximal cut vector to level j for historical reasons.

Suppose now that the system (C, ϕ) allows a modular decomposition of the form

$$\phi(\mathbf{x}) = \psi(\boldsymbol{\chi}(\mathbf{x})) = \psi(\chi_1(\mathbf{x}_{A_1}), \dots, \chi_r(\mathbf{x}_{A_r})), \quad (3)$$

where A_1, \dots, A_r is a partition of C , \mathbf{x}_{A_k} is the vector with components $x_i, i \in A_k, k = 1, \dots, r$, ψ is a structure function called the organizing structure function, and χ_1, \dots, χ_r are structure functions called the modular structure functions. It is then possible to construct bounds for p_ϕ^j by combining bounds for the availabilities for the organizing structure and bounds for the availabilities of the modules, based on the sets of minimal path and cut vectors for ψ and χ_1, \dots, χ_r respectively. This problem has been considered by Bodin (1970) in the binary case, and by Butler (1982) and Funnemark and Natvig (1985) in the multistate case, with some refinements in Natvig (1986) and Natvig (1993). All the bounds constructed in these papers build on the common assumption that the processes $\{\mathbf{X}_{A_k}(t), t \in I\}$ are stochastically independent in I , and we will stick to this assumption throughout this paper. For such bounds to be useful, they must be shown to be advantageous in comparison with the basic bounds based on minimal path and cut vectors for ϕ , as given in Funnemark and Natvig (1985). A proper understanding of the relationship between the minimal path and cut vectors for the structure functions ϕ, ψ and χ_1, \dots, χ_r is a necessary basis for such a comparison. It turns out that the comparisons made by Butler (1982) are based on an inadequate analysis of this relationship. These shortcomings are inherited by Funnemark and Natvig (1985), who build on the work of Butler (1982). As a result, we have found it necessary to consider some of the results of these papers again.

In this paper, we start out in section 2 by introducing some necessary notation and reviewing the bounds that are relevant to our analysis. In section 3 we analyse in depth the relationship between the minimal path and cut vectors for the different structure functions involved in a modular decomposition. Based on this analysis, we establish assumptions which are shown in section 5 to ensure that the results of Butler (1982) on lower bounds based on minimal cut vectors are valid. In addition, we extend one of these results, valid when the component processes are independent in I , from the availability at a fixed point of time to the availability in an interval. In section 4 we reprove the results on lower bounds based on minimal path vectors whose proofs in Funnemark and Natvig (1985) rest on the inaccurate results of Butler (1982). In section 6 we discuss combinations of different kinds of bounds, while we consider bounds based on refining the modular decompositions in section 7.

2 Notation and basic bounds

For the sake of readability, we will try to keep the notation as simple as possible, and hence deviate to some extent from the notation used in Funnemark and Natvig (1985).

The full notation for the modular decomposition defined by (3) is

$$(\psi, (A_1, \chi_1), \dots, (A_r, \chi_r)),$$

but with the partition A_1, \dots, A_r implicitly understood, this will often be referred to as (ψ, χ) . The range of the structure functions χ_k may be proper subsets of S , denoted by S_{χ_k} , always assumed to contain 0 and M . For any $j \in S_{\chi_k}, j < M$, we define $j^+(S_{\chi_k}) = \min\{j' | j' \in S_{\chi_k}, j' > j\}$. When the state space S_{χ_k} is clear from the context, we often simplify this, and write $j^+(S_{\chi_k}) = j^+$. We let arbitrarily $M^+ = M + 1$.

We consider availabilities and unavailabilities in a fixed interval I , and do not refer to this interval in the notation for the bounds, or in the formulation of the results, unless explicit mention of the interval is needed. This is so in theorem 3, dealing with two different intervals simultaneously. It is also referred to the interval in the special case $I = [t, t]$ in the formulation of theorem 4.

The system availabilities to the different levels are defined in (2) and are collected in the vector

$$\mathbf{p}_\phi = (p_\phi^1, \dots, p_\phi^M).$$

We use the same notation for the availabilities of the structure functions of the modules and for vectors whose components are these availabilities. These vectors are collected to form the vector

$$\mathbf{p}_\chi = (\mathbf{p}_{\chi_1}, \dots, \mathbf{p}_{\chi_r}).$$

This notation is extended down through an increasingly refined hierarchy of modular decompositions to end up at the component level, with the component availabilities $p_{X_i}^j$ to the different levels $j \in S_i$, defined in (1), collected in the vectors \mathbf{p}_{X_i} , again collected in

$$\mathbf{p}_\mathbf{x} = (\mathbf{p}_{X_1}, \dots, \mathbf{p}_{X_n}).$$

Unavailabilities, defined by replacing p by q and \geq by $<$ in the definitions for availabilities, are treated similarly.

In order to minimize the notation, throughout this paper we consider

availabilities and unavailabilities for ϕ and ψ to a fixed level j , representative of any level of interest, and we often drop this j in the notation, especially in the notation related to minimal path and cut vectors. For the modules, availabilities and unavailabilities to every level is relevant, but are implicitly included by the notation \mathbf{p}_{χ_k} and \mathbf{p}_{χ} .

We use the corresponding notation for bounds, e.g.

$$l_{\chi_k}^{*j}, \mathbf{l}_{\chi_k}^* = (l_{\chi_k}^{*1}, \dots, l_{\chi_k}^{*M}) \text{ and } \mathbf{l}_{\chi}^* = (\mathbf{l}_{\chi_1}^*, \dots, \mathbf{l}_{\chi_r}^*).$$

Here we refer to the lower bound that for the system (C, ϕ) is defined in terms of the minimal cut vectors $\mathbf{z}^m, m = 1, \dots, M_c$ to level j (see Butler (1982) for the case $I = [t, t]$ and Funnemark and Natvig (1985) for the general case) by

$$\begin{aligned} l_{\phi}^{*j} &= \prod_{m=1}^{M_c} P((\cup_{i=1}^n (X_i(t) > z_i^m)) \text{ for all } t \in \tau(I)) = \\ &\prod_{m=1}^{M_c} P([\max_{1 \leq i \leq n} (X_i(t) - z_i^m)] > 0 \text{ for all } t \in \tau(I)). \end{aligned} \quad (4)$$

This is a lower bound for p_{ϕ}^j if the component processes are associated in I , see Esary and Proschan (1970), for a definition of the concept of association in I .

The basic lower bound using the minimal path vectors $\mathbf{y}^m, m = 1, \dots, M_p$ to level j is given by Funnemark and Natvig (1985) as

$$\begin{aligned} l_{\phi}''^j &= \max_{1 \leq m \leq M_p} P(\cap_{i=1}^n (X_i(t) \geq y_i^m \text{ for all } t \in \tau(I))) = \\ &\max_{1 \leq m \leq M_p} (P(\cap_{i=1}^n (\check{X}_i \geq y_i^m))), \end{aligned} \quad (5)$$

where by definition

$$\check{X}_i = \min_{t \in \tau(I)} (X_i(t)). \quad (6)$$

This lower bound is valid regardless of the joint distribution of the component processes.

The bounds (4) and (5) are not determined by the component availabilities (1) alone. The bound (5) is determined by the specification of the joint

distribution of $\check{\mathbf{X}} = (\check{X}_1, \dots, \check{X}_n)$, and requires the calculation of joint probabilities in this distribution. The calculation of (4) is even more demanding. Such calculations are unrealistic in many cases, and we are then left with simplified versions. The bound (4) takes the following form, based on independent component processes (see Butler (1982) and Funnemark and Natvig (1985))

$$l_{\phi}^{**j}(\mathbf{p}\mathbf{x}) = \prod_{m=1}^{M_c} \prod_{i=1}^n P(\check{X}_i > z_i^m) = \prod_{m=1}^{M_c} \prod_{i=1}^n p_i^{z_i^m+1}, \quad (7)$$

where we have used the "ip"-operator \prod defined by

$$\prod_{i=1}^n p_i = 1 - \prod_{i=1}^n (1 - p_i).$$

Note that if I is reduced to a single point $[t, t]$, and if the component states are independent at t , then (4) and (7) coincide.

A correspondingly simplified version of (5) gives a valid lower bound under the assumption of associated component processes in I ,

$$l_{\phi}^{'j}(\mathbf{p}\mathbf{x}) = \max_{1 \leq m \leq M_p} \left(\prod_{i=1}^n P(\check{X}_i \geq y_i^m) \right) = \max_{1 \leq m \leq M_p} \left(\prod_{i=1}^n p_i^{y_i^m} \right). \quad (8)$$

If the component processes are independent in I , then (8) is identical with (5).

Corresponding to these bounds for availabilities, there are bounds for unavailabilities. To present two examples, which also indicate what kind of notation is used in general, we have, parallel to (5) and (8),

$$\begin{aligned} \bar{l}_{\phi}^{'j} &= \max_{1 \leq m \leq M_c} P(\cap_{i=1}^n (X_i(t) \leq z_i^m) \text{ for all } t \in \tau(I)) = \\ &= \max_{1 \leq m \leq M_c} (P(\cap_{i=1}^n (\hat{X}_i \leq z_i^m))) \end{aligned} \quad (9)$$

and

$$\bar{l}_{\phi}^{'j}(\mathbf{q}\mathbf{x}) = \max_{1 \leq m \leq M_c} \left(\prod_{i=1}^n P(\hat{X}_i \leq z_i^m) \right) = \max_{1 \leq m \leq M_c} \left(\prod_{i=1}^n q_i^{z_i^m+1} \right), \quad (10)$$

where, similar to (6),

$$\hat{X}_i = \max_{t \in \tau(I)} (X_i(t)). \quad (11)$$

Our main focus in this paper is on the availability bounds (7) and (8) that are explicit functions of the vector $\mathbf{p}_\mathbf{x}$. However, we will also treat the bounds (4) and (5), both for the sake of completeness, and because this is a natural part of the mathematical development.

The principle behind using modular decompositions in connection with availability bounds is that we replace the vector of availabilities for modules by a corresponding vector of bounds. For instance, we replace \mathbf{p}_χ in the lower bound $l_\psi^{**j}(\mathbf{p}_\chi)$ for $p_\psi^j = p_\phi^j$ by \mathbf{l}_χ^{**} . In section 4 we will prove (see theorem 2), as stated in Funnemark and Natvig (1985), that

$$l_\psi^{'j}(\mathbf{l}_\chi'(\mathbf{p}_\mathbf{x})) = l_\phi^{'j}(\mathbf{p}_\mathbf{x}), \quad (12)$$

and use this to prove an inequality concerning upper bounds (see theorem 3), stated in Funnemark and Natvig (1985) and Natvig (1986). In section 5 we will prove (see theorem 5) that under certain additional assumptions (see theorem 1)

$$l_\psi^{**j}(\mathbf{l}_\chi^{**}(\mathbf{p}_\mathbf{x})) \geq l_\phi^{**j}(\mathbf{p}_\mathbf{x}). \quad (13)$$

This inequality was claimed to be true by Butler (1982) and Funnemark and Natvig (1985), but only for the case of $I = [t, t]$, and without realising the need for additional assumptions.

3 Minimal path vectors (mpvs) and cut vectors (mcvs) for the system, the organizing structure and the modules

From now on we use the abbreviation mpv for "minimal path vector" and mcv for "minimal cut vector". For mpvs $\mathbf{u} = (u_1, \dots, u_r)$ and mcvs $\mathbf{v} = (v_1, \dots, v_r)$ for ψ to level j we will have to consider mpvs for χ_k to level u_k and mcvs for χ_k to level $v_k^+(S_{\chi_k})$, $k = 1, \dots, r$. Since we may have $u_k = 0$ and $v_k = M$, we must deal with path vectors for χ_k to level 0, $(0, \dots, 0)$ obviously being the only minimal one, and cut vectors for χ_k to level $M+1$, (M, \dots, M) obviously being the only minimal one.

The following lemma is a key result for the analysis of the availability bounds based on modular decomposition.

Lemma 1 Let (ψ, χ) be a modular decomposition of (C, ϕ) , defined as in (3).

a) The mpvs for ϕ, ψ, χ are related as follows:

- (i) Let \mathbf{y} be an mpv for ϕ to level j . For each $k = 1, \dots, r$ let $j_k = \chi_k(\mathbf{y}_{A_k})$. Then there exists an mpv \mathbf{u} for ψ to level j with $\mathbf{u} \leq (j_1, \dots, j_r)$, and for each such \mathbf{u} and each $k = 1, \dots, r$, \mathbf{y}_{A_k} is an mpv for χ_k to level u_k .
- (ii) Suppose that for each $k = 1, \dots, r$, $u_k \in S_{\chi_k}$, every mpv \mathbf{y}_{A_k} to level u_k satisfies $\chi_k(\mathbf{y}_{A_k}) = u_k$. Then the vector \mathbf{u} of (i) is unique.
- (iii) Make the same assumption as in (ii). Let \mathbf{u} be an mpv for ψ to level j . For each $k = 1, \dots, r$, let \mathbf{y}_{A_k} be an mpv for χ_k to level u_k . Then $\mathbf{y} = (\mathbf{y}_{A_1}, \dots, \mathbf{y}_{A_r})$ is an mpv for ϕ to level j .

b) The mcvs for ϕ, ψ, χ are related as follows:

- (i) Let \mathbf{z} be an mcv for ϕ to level j . For each $k = 1, \dots, r$ let $j_k = \chi_k(\mathbf{z}_{A_k})$. Then there exists an mcv \mathbf{v} for ψ to level j with $\mathbf{v} \geq (j_1, \dots, j_r)$, and for each such \mathbf{v} and each $k = 1, \dots, r$, \mathbf{z}_{A_k} is an mcv for χ_k to level $v_k^+ = v_k^+(S_{\chi_k})$.
- (ii) Suppose that for each $k = 1, \dots, r$, $v_k \in S_{\chi_k}$, every mcv \mathbf{z}_{A_k} to level v_k^+ satisfies $\chi_k(\mathbf{z}_{A_k}) = v_k$. Then the vector \mathbf{v} of (i) is unique.
- (iii) Make the same assumption as in (ii). Let \mathbf{v} be an mcv for ψ to level j . For each $k = 1, \dots, r$, let \mathbf{z}_{A_k} be an mcv for χ_k to level v_k^+ . Then $\mathbf{z} = (\mathbf{z}_{A_1}, \dots, \mathbf{z}_{A_r})$ is an mcv for ϕ to level j .

Proof: To prove (i) of part a), note that clearly (j_1, \dots, j_r) is a path vector, but not necessarily minimal, for ψ to level j . Let \mathbf{u} be any mpv to level j for ψ , satisfying $\mathbf{u} \leq (j_1, \dots, j_r)$. Then for each $k = 1, \dots, r$ \mathbf{y}_{A_k} is a path vector for χ_k to level u_k . We claim that it is in fact an mpv. To see this, assume for simplicity that $k = 1$. If $\mathbf{y}_{A_1} = (0, \dots, 0)$, then $u_1 = 0$, and the claim is satisfied. Otherwise, choose an arbitrary $i \in A_1$ for which $y_i > 0$. Let y_i be replaced by some $y'_i < y_i$, to give rise to the modified vectors $\mathbf{y}'_{A_1}, \mathbf{y}'$, leaving all components except y_i unchanged. We then have

$$\psi(\mathbf{u}) \geq j > \phi(\mathbf{y}') =$$

$$\psi(\chi_1(\mathbf{y}'_{A_1}), \chi_2(\mathbf{y}_{A_2}), \dots, \chi_r(\mathbf{y}_{A_r})) \geq \psi(\chi_1(\mathbf{y}'_{A_1}), u_2, \dots, u_r).$$

This means that we must have $u_1 > \chi_1(\mathbf{y}'_{A_1})$, proving that \mathbf{y}_{A_1} is an mpv for χ_1 to level u_1 . Under the assumption of (ii), the vector \mathbf{u} satisfies $u_k = \chi_k(\mathbf{y}_{A_k}) = j_k$. Hence, $\mathbf{u} = (j_1, \dots, j_r)$. To prove (iii), let \mathbf{u} and \mathbf{y} be as stated in (iii). Then clearly \mathbf{y} is a path vector for ϕ to level j . Choose $y'_i < y_i$ for some i . For simplicity, we may assume $i \in A_1$. Let \mathbf{y}' and \mathbf{y}'_{A_1} represent the corresponding adjustments of \mathbf{y} and \mathbf{y}_{A_1} , as above. Then $\chi_1(\mathbf{y}'_{A_1}) = u'_1 < u_1$. For $k \neq 1$, by assumption, $\chi_k(\mathbf{y}_{A_k}) = u_k$. Hence,

$$\phi(\mathbf{y}') = \psi(\chi_1(\mathbf{y}'_{A_1}), \chi_2(\mathbf{y}_{A_2}), \dots, \chi_r(\mathbf{y}_{A_r})) = \psi(u'_1, u_2, \dots, u_r) < j,$$

proving that \mathbf{y} is an mpv to level j . The proof of part b) is similar.

Remark 1 *The condition in (ii) of part a) is sufficient to obtain uniqueness in the determination of an appropriate \mathbf{u} , but the vector \mathbf{u} may be uniquely determined also without it. An example is provided by assuming that χ_k satisfies the condition for $k = 2, \dots, r$, while χ_1 does not. Then we have $u_k = j_k$ for $k = 2, \dots, r$, and it follows that u_1 is uniquely determined, i.e. the conclusion of (ii) of part a) holds. However, we are not able to prove that the conclusion of (iii) holds in these circumstances.*

Butler (1982) tries to characterize the mcvs for ϕ by means of binary structure functions related to the mcvs for ψ . In our notation, the set of vectors that he claims to be the distinct mcvs for ϕ , without any additional assumption on the structure functions χ_k , is in fact identical to the set that arises from the construction in (iii), part b), of lemma 1. The following example shows that this set may in fact contain replicates of certain cut vectors, and also cut vectors that are not minimal. We demonstrate by this example that without the assumption of (ii) of part a) and b) respectively, the construction in (i) is not necessarily unique, and vectors \mathbf{y}, \mathbf{z} of the form given in (iii) are not necessarily mpvs respectively mcvs.

Example. Let $C = \{1, 2, 3, 4\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, $S = \{0, \dots, 3\}$, $S_i = \{0, 1, 3\}$, $i = 1, \dots, 4$. Let the mpvs for ψ be $(3, 3)$ to level 3, $(3, 1)$, $(1, 3)$, $(2, 2)$ to level 2 and $(1, 1)$ to level 1. Let the mpvs for $\chi_1 = \chi_2$ be $(3, 1)$, $(1, 3)$ to level 3, $(3, 1)$, $(0, 3)$ to level 2, and $(1, 1)$, $(3, 0)$ to level 1. Hence, $(3, 1)$ is an mpv both to level 3 and level 2 for χ_k , $k = 1, 2$. Following the procedure of (iii) of lemma 1, part a), with $\mathbf{u} = (2, 2)$, we obtain a vector $(3, 1, 3, 1)$

which is not an mpv to level 2, since e.g. $(3, 1, 1, 1)$ is a path vector to level 2. Hence, the procedure of part (iii) does not always give an mpv. On the other hand, $(1, 1, 1, 1)$ is an mcv to level 2 for ϕ . Following the procedure of (i) of lemma 1, part b), we see that $\chi_1((1, 1)) = \chi_2((1, 1)) = 1$. We seek an mcv $\mathbf{v} \geq (1, 1)$ for ψ to level 2, and observe that both $(2, 1)$ and $(1, 2)$ meet this requirement. Hence, the procedure of (i) of lemma 1, part b) does not always give a unique mcv \mathbf{v} .

By duality, this example also shows that in general the procedure of (i) of lemma 1, part a), does not necessarily give a unique mpv \mathbf{u} for ψ , and that a vector \mathbf{z} constructed as in (iii) of lemma 1, part b), is not necessarily an mpv for ϕ .

It is convenient to enumerate the different sets of mpvs and mcvs. This is done in the following definition, which also introduces some more useful notation.

Definition 1 *Let the mpvs to level j for ψ and ϕ respectively be*

$$\{\mathbf{u}^l : l = 1, \dots, L_p\} \text{ and } \{\mathbf{y}^m : m = 1, \dots, M_p\}.$$

Also, for each $\mathbf{u}^l = (u_1^l, \dots, u_r^l)$ and each $k = 1, \dots, r$, let the mpvs for χ_k to level u_k^l be

$$\{\mathbf{y}_{A_k}^{l,s} : s \in \{1, 2, \dots, e_{l,k}\} = E_{l,k}\}.$$

We denote the components of $\mathbf{y}_{A_k}^{l,s}$ by $y_i^{l,k,s}, i \in A_k$.

Similarly, let the mcvs to level j for ψ and ϕ respectively be

$$\{\mathbf{v}^l : l = 1, \dots, L_c\} \text{ and } \{\mathbf{z}^m : m = 1, \dots, M_c\}.$$

Also, for each $\mathbf{v}^l = (v_1^l, \dots, v_r^l)$ and each $k = 1, \dots, r$, let the mcvs for χ_k to level v_k^l be $\{\mathbf{z}_{A_k}^{l,s} : s \in \{1, 2, \dots, b_{l,k}\} = B_{l,k}\}$.

We denote the components of $\mathbf{z}_{A_k}^{l,s}$ by $z_i^{l,k,s}, i \in A_k$.

With these definitions we can now rephrase in the following theorem the main content of lemma 1, providing a sufficient, operational condition which ensures that there is a one-to-one correspondance between, on the one hand, mpvs \mathbf{y} for ϕ to level j , and, on the other hand, mpvs \mathbf{u} for ψ to level j and corresponding mpvs \mathbf{y}_{A_k} for χ_k to level $u_k, k = 1, \dots, r$. The theorem also provides a sufficient, operational condition ensuring that there is a one-to-one correspondance between, on the one hand, mcvs \mathbf{z} for ϕ to level j , and,

on the other hand, mcvs \mathbf{v} for ψ to level j and corresponding mcvs \mathbf{z}_{A_k} for χ_k to level $v_k^+, k = 1, \dots, r$. In turn, this one-to-one correspondance is used in section 5 to prove the inequality (13).

Theorem 1 *Let (ψ, χ) be a modular decomposition of (C, ϕ) , as defined by (3). Suppose the following condition is satisfied:*

- (i) *For each $k = 1, \dots, r, u_k \in S_{\chi_k}$, every mpv \mathbf{y}_{A_k} to level u_k satisfies $\chi_k(\mathbf{y}_{A_k}) = u_k$.*

Then, with the notation of definition 1:

- (ii) *For each $l \in \{1, 2, \dots, L_p\}$ and each $\mathbf{s} \in E_{l,1} \times \dots \times E_{l,r}$, there exists $m \in \{1, 2, \dots, M_p\}$ such that*

$$\mathbf{y}^m = (\mathbf{y}_{A_1}^{l,s_1}, \dots, \mathbf{y}_{A_r}^{l,s_r}).$$

Conversely, each \mathbf{y}^m can be written uniquely in this way.

Similarly, suppose the following condition is satisfied:

- (iii) *For each $k = 1, \dots, r, v_k \in S_{\chi_k}$, every mcv \mathbf{z}_{A_k} to level v_k^+ satisfies $\chi_k(\mathbf{z}_{A_k}) = v_k$.*

Then, with the notation of definition 1:

- (iv) *For each $l \in \{1, 2, \dots, L_c\}$ and each $\mathbf{s} \in B_{l,1} \times \dots \times B_{l,r}$, there exists $m \in \{1, 2, \dots, M_c\}$ such that*

$$\mathbf{z}^m = (\mathbf{z}_{A_1}^{l,s_1}, \dots, \mathbf{z}_{A_r}^{l,s_r}).$$

Conversely, each \mathbf{z}^m can be written uniquely in this way.

With the notation of definition 1 we can now also write the lower bound based on a modular decomposition introduced in (12) as

$$l_{\psi}^{'j}(\mathbf{l}'_{\chi}(\mathbf{p}\mathbf{x})) = \max_{1 \leq l \leq L_p} \prod_{k=1}^r [\max_{s \in E_{l,k}} \prod_{i \in A_k} p_i^{y_i^{l,k,s}}] \quad (14)$$

The lower bound based on a modular decomposition introduced in (13) can be written

$$l_{\psi}^{**j}(\mathbf{l}^{**}_{\chi}(\mathbf{p}\mathbf{x})) = \prod_{l=1}^{L_c} \prod_{k=1}^r \prod_{s \in B_{l,k}} \prod_{i \in A_k} p_i^{z_i^{l,k,s}+1}. \quad (15)$$

4 Lower bounds for availabilities based on minimal path vectors and for unavailabilities based on minimal cut vectors, and corresponding upper bounds

The following theorem shows that the lower bounds (5) and (8) are unchanged under a modular decomposition.

Theorem 2 *Let (ψ, χ) be a modular decomposition of (C, ϕ) as in (3). Let $j \in S$ be arbitrary. Comparing lower bounds for p_ϕ^j based on mpvs, we then have*

$$l_\phi'' = l'_\psi(l''_\chi) \quad (16)$$

and

$$l'_\phi(\mathbf{px}) = l'_\psi(l'_\chi(\mathbf{px})) \quad (17)$$

Analogous results are valid for the corresponding lower unavailability bounds.

Proof: Let \mathbf{y} be an mpv for ϕ to level j for which the maximum in the definition of l_ϕ'' is obtained. Choose an mpv \mathbf{u}^l for ψ to level j such that $\chi_k(\mathbf{y}_{A_k}) \geq u_k^l, k = 1, \dots, r$. By (i) of lemma 1, part a), using the notation of definition 1, \mathbf{y} is of the form $\mathbf{y} = (\mathbf{y}_{A_1}^{l,s_1}, \dots, \mathbf{y}_{A_r}^{l,s_r})$ for some $\mathbf{s} = (s_1, \dots, s_r) \in E_{l,1} \times \dots \times E_{l,r}$. Due to the independence of the modules we have

$$l_\phi'' = \prod_{k=1}^r P(X_i(t) \geq y_i^{l,k,s_k} \text{ for all } i \in A_k, t \in \tau(I)) \leq \max_{1 \leq l \leq L_p} \prod_{k=1}^r [\max_{s_k \in E_{l,k}} P(X_i(t) \geq y_i^{l,k,s_k} \text{ for all } i \in A_k, t \in \tau(I))] = l'_\psi(l''_\chi). \quad (18)$$

To prove the opposite inequality, choose \mathbf{u}^l and $(\mathbf{y}_{A_1}^{l,s_1}, \dots, \mathbf{y}_{A_r}^{l,s_r})$ for which the right hand side of the inequality (18) is attained. Then $\mathbf{y} = (\mathbf{y}_{A_1}^{l,s_1}, \dots, \mathbf{y}_{A_r}^{l,s_r})$ is a path vector for ϕ to level j . In case it is not minimal, choose an mpv $\mathbf{y}' \leq \mathbf{y}$. Then

$$l_\phi'' \geq P(X_i(t) \geq y'_i \text{ for all } i = 1, \dots, n, t \in \tau(I)) =$$

$$\prod_{k=1}^r P(X_i(t) \geq y'_i \text{ for all } i \in A_k, t \in \tau(I)) \geq \prod_{k=1}^r P(X_i(t) \geq y_i^{l,k,s_k} \text{ for all } i \in A_k, t \in \tau(I)) = l'_\psi(\mathbf{l}''_\chi),$$

where the last inequality follows since $y_i^{l,k,s_k} \geq y'_i$ for all $i \in A_k, k = 1, \dots, r$. Hence, (16) is proved.

If the component processes $X_i, i = 1, \dots, n$, are independent in I , then (17) is equivalent to (16), so (17) is true in this case. This means that we have

$$\max_{1 \leq l \leq L_p} \prod_{k=1}^r [\max_{\mathbf{s} \in E_{l,1} \times \dots \times E_{l,r}} \prod_{i \in A_k} P(X_i(t) \geq y_i^{l,k,s_k} \text{ for all } t \in \tau(I)) = \max_{1 \leq m \leq M_p} \prod_{i=1}^n P(X_i(t) \geq y_i^m \text{ for all } t \in \tau(I)) \quad (19)$$

We need to prove that (19) is true in general, i.e. for a system with an arbitrary dependence structure between the component processes. But this follows by considering a different system which has the same structure function and the same marginal distributions for the component processes as the original one, but in which the component processes are independent. This completes the proof.

This theorem shows that nothing is neither gained nor lost by using a modular decomposition in connection with the l' - and l'' -bounds in terms of closeness to p_ϕ^j . However, the modular decomposition may be advantageous from the computational point of view. The theorem is also stated in Funnemark and Natvig (1985), but the proof uses the inadequate characterization of the mpvs of ϕ of Butler (1982). The same holds true for the comparison of upper availability bounds, originally presented in Funnemark and Natvig (1985) and Natvig (1986), given in theorem 3 below. Note that the upper bound $u_\phi''(I)$ appearing in the theorem, is based on taking the minimum over the factors appearing in the lower bound (4) rather than their product.

Theorem 3 *Let (ψ, χ) be a modular decomposition of (C, ϕ) as in (3). Let $j \in S$ be arbitrary, and consider the following upper bound for p_ϕ^j :*

$$u_\phi''(I) = \min_{1 \leq m \leq M_c} P([\max_{1 \leq i \leq n} (X_i(t) - z_i^m)] > 0 \text{ for all } t \in \tau(I)).$$

We then have

$$u_\phi''(I) \leq \inf_{t \in \tau(I)} (1 - \bar{l}''_\phi([t, t])) = \inf_{t \in \tau(I)} (1 - \bar{l}''_\psi(\bar{\mathbf{l}}''_\chi([t, t]))).$$

The corresponding result is valid for the upper bounds for unavailabilities to level j based on mpvs.

Proof: Choose an arbitrary $t \in \tau(I)$. For any mcv \mathbf{z} of ϕ to level j , we have

$$P([\max_{1 \leq i \leq n}(X_i(t) - z_i)] > 0) = P(\cup_{i=1}^n (X_i(t) > z_i)) = 1 - P(\cap_{i=1}^n (X_i(t) \leq z_i)).$$

Minimizing this over mcvs \mathbf{z} on both sides and using (9) yields

$$u''_\phi([t, t]) = 1 - \bar{l}''_\phi([t, t]).$$

We obviously have $u''_\phi(I) \leq u''_\phi([t, t])$. Hence, the stated inequality follows by taking the infimum over all $t \in \tau(I)$. The subsequent equality follows by using the unavailability part of theorem 2.

5 Lower availability bounds based on minimal cut vectors

In order to deal with the l^* and l^{**} bounds of (4) and (7), we need the following lemma:

Lemma 2 *Let $P^{k,s}, k = 1, \dots, r, s \in B_k = \{1, \dots, b_k\}$ be real numbers between 0 and 1. Then*

$$\prod_{\mathbf{s} \in B_1 \times \dots \times B_r} \prod_{k=1}^r P^{k,s_k} \leq \prod_{k=1}^r \prod_{s \in B_k} P^{k,s}.$$

Proof: If $b_k = 1$ for each $k = 1, \dots, r$, the lemma is obviously true, since then both sides equal $\prod_{k=1}^r P^{k,1}$. Hence, the lemma is true when $N = \sum_{k=1}^r b_k - r = 0$. We prove the lemma by induction on N . We need the following inequality, valid for $p, q, w \in [0, 1]$:

$$pq \amalg w - (p \amalg w)(q \amalg w) \geq 0. \quad (20)$$

The inequality follows, since

$$pq \amalg w - (p \amalg w)(q \amalg w) = (pq + w - pqw) - q(p + w - pw) - w(1 - q)(p \amalg w) = w(1 - q) - w(1 - q)(p \amalg w) \geq 0.$$

With obvious interpretation of p, q, w , this covers the special case $r = 2, b_1 = 2, b_2 = 1$ of the lemma. With the interpretation $w = p^{2,1} \amalg \dots \amalg p^{r,1}$, (20)

also covers the case $b_1 = 2, b_2 = \dots = b_r = 1, r$ arbitrary. By symmetry, the lemma is true for $N = 1$. Now assume the lemma to be true for some $N \geq 1$, and consider the case $N + 1$. We may assume that $b_1 \geq 2$. Define $p = \prod_{s=1}^{b_1-1} P^{1,s}, q = P^{1,b_1}, w = \coprod_{k=2}^r \prod_{s \in B_k} P^{k,s}$. Using (20) and the induction hypothesis we then have

$$\begin{aligned} \coprod_{k=1}^r \prod_{s \in B_k} P^{k,s} &= pq \coprod w \geq (p \coprod w)(q \coprod w) \geq \\ &(\prod_{\mathbf{s} \in (B_1 - \{b_1\}) \times B_2 \times \dots \times B_r} \coprod_{k=1}^r P^{k,s_k}) (\prod_{\mathbf{s} \in B_2 \times \dots \times B_r} P^{1,b_1} \coprod (\coprod_{k=2}^r P^{k,s_k})) = \\ &\prod_{\mathbf{s} \in B_1 \times \dots \times B_r} \coprod_{k=1}^r P^{k,s_k}, \end{aligned}$$

proving the lemma.

In our analysis of the behaviour of the l^* and l^{**} bounds, we have to start with the special case of an interval of the form $[t, t]$ for an arbitrary $t \in \tau(I)$. For this special case we introduce the following notation, which simplifies the mathematical expressions. Recalling from definition 1 that $z_i^{l,k,s}$ are the components of the mcvs $\mathbf{z}_{A_k}^{l,s}, s \in B_{l,k}$, of χ_k to level $v_k^l +$, define

$$P^{l,k,s} = P(\cup_{i \in A_k} (X_i(t) > z_i^{l,k,s})), l = 1, \dots, L_c, k = 1, \dots, r, s \in B_{l,k}. \quad (21)$$

Using this notation and replacing ϕ by χ_k in (4), we then have

$$l_{\chi_k}^{*(v_k^l +)} = \prod_{s \in B_{l,k}} P^{l,k,s} \quad (22)$$

which is a component in the vector \mathbf{l}_{χ}^* .

Theorem 4 *Let $t \in \tau(I)$. If the modular decomposition (ψ, χ) of (C, ϕ) has the property described in (iv) of theorem 1, and in particular if the modular structure functions $\chi_k, k = 1, \dots, r$, satisfy the condition (iii) of that theorem, then*

$$l_{\phi}^*([t, t]) \leq l_{\psi}^{**}(\mathbf{l}_{\chi}^*)([t, t]).$$

Proof: Using the property of (iv) of theorem 1, and then the independence of the modules and the simplifying notation (21), we have

$$\begin{aligned} l_{\phi}^*([t, t]) &= \prod_{l=1}^{L_c} \prod_{\mathbf{s} \in B_{l,1} \times \dots \times B_{l,r}} P(\cup_{k=1}^r \cup_{i \in A_k} (X_i(t) > z_i^{l,k,s_k})) = \\ &\prod_{l=1}^{L_c} \prod_{\mathbf{s} \in B_{l,1} \times \dots \times B_{l,r}} \prod_{k=1}^r P^{l,k,s_k}. \end{aligned} \quad (23)$$

We now use lemma 2 to each

$$\prod_{\mathbf{s} \in B_{l,1} \times \dots \times B_{l,r}} \prod_{k=1}^r P^{l,k,s_k}, l = 1, \dots, L_c.$$

of (23), and then use (22) to obtain

$$l_\phi^*([t, t]) \leq \prod_{l=1}^{L_c} \prod_{k=1}^r \prod_{s \in B_{l,k}} P^{l,k,s} \quad (24)$$

$$= \prod_{l=1}^{L_c} \prod_{k=1}^r l_{\chi_k}^{*(v_k^{l,+})} = l_\psi^{**}(\mathbf{l}_\chi^*([t, t])),$$

where the last equality follows by replacing ϕ by ψ in (7). This completes the proof.

By Funnemark and Natvig (1985), theorem 4 provides valid lower bounds for p_ϕ^j if the component states of each module, and hence all component states, at time t are sets of associated random variables.

Attempting to extend equation (23) to the case of a general interval I in place of $[t, t]$ leads to the inequality (cf. (4))

$$\begin{aligned} & P([\max_{1 \leq k \leq r} \max_{i \in A_k} (X_i(t) - z_i^{l,k,s})] > 0 \text{ for all } t \in \tau(I)) \geq \\ & P(\cup_{k=1}^r ([\max_{i \in A_k} (X_i(t) - z_i^{l,k,s})] > 0 \text{ for all } t \in \tau(I))) = \\ & \prod_{k=1}^r P([\max_{i \in A_k} (X_i(t) - z_i^{l,k,s})] > 0 \text{ for all } t \in \tau(I)). \end{aligned}$$

Since this inequality has the wrong direction, we are not able to generalize theorem 4 to the case of a general interval. Roughly speaking, the explanation for this is that the expression on the left hand side of the inequality allows different components to prevent the cut vector from sabotaging the system at different times, while the expression on the right hand side only allows different components within a single module to do so. However, specializing to the bounds based on independent components, we are finally able to obtain an extension to an arbitrary interval I :

Theorem 5 *Let I be an arbitrary interval, and assume that the component processes are independent in I . If the modular decomposition (ψ, χ) of (C, ϕ) has the property described in (iv) of theorem 1, and in particular if the modular structure functions $\chi_k, k = 1, \dots, r$, satisfy the condition (iii) of that theorem, then*

$$l_\phi^{**}(\mathbf{p}_\mathbf{x}) \leq l_\psi^{**}(\mathbf{l}_\chi^{**}(\mathbf{p}_\mathbf{x}))$$

Proof: Consider first the case of a degenerate interval $[t, t]$. Since we are considering availabilities at a single point of time, and since the component states at this point of time are independent, the l^* and l^{**} bounds coincide. Hence, the result follows by theorem 4. Comparing (23) and (24), this means that

$$\prod_{l=1}^{L_c} \prod_{\mathbf{s} \in B_{l,1} \times \dots \times B_{l,r}} \prod_{k=1}^r \prod_{i \in A_k} p_i^{l,k,s_k} \leq \prod_{l=1}^{L_c} \prod_{k=1}^r \prod_{\mathbf{s} \in B_{l,k}} \prod_{i \in A_k} p_i^{l,k,s}, \quad (25)$$

where we have defined

$$p_i^{l,k,s} = P(X_i(t) > z_i^{l,k,s}).$$

But the inequality (25) remains valid if $p_i^{l,k,s}$ is redefined to mean

$$p_i^{l,k,s} = P(X_i(t) > z_i^{l,k,s} \text{ for all } t \in \tau(I)) = p_i^{z_i^{l,k,s}+1}.$$

Using the property of (iv) of theorem 1 and equation (7), the left hand side of the inequality (25) then becomes $l_\phi^{**j}(\mathbf{p}_\mathbf{x})$, while the right hand side equals $l_\psi^{**j}(\mathbf{l}_\chi^{**}(\mathbf{p}_\mathbf{x}))$ by equation (15). This completes the proof.

In this section for simplicity we have only considered lower availability bounds, although corresponding results for unavailability bounds, involving condition (i) and property (ii) of theorem 1, can be given.

6 Combination of availability bounds and comparisons based on monotonicity

Suppose first that the component performance processes $X_i(\cdot), i = 1, \dots, n$, are independent in I . It depends on the structure function ϕ and on $\mathbf{p}_\mathbf{x}$ whether l_ϕ^{**j} or $l_\phi'^j$ is the better bound. Moreover, contrary to intuition, the l_ϕ^{**j} -bound is not necessarily non-increasing in j . Hence, the best possible bounds based on (7) and (8) are obtained by maximization, and is defined as follows (see Funnemark and Natvig (1985)):

$$B_\phi^{*j}(\mathbf{p}_\mathbf{x}) = \max_{j' \geq j} \max(l_\phi^{**j'}(\mathbf{p}_\mathbf{x}), l_\phi'^{j'}(\mathbf{p}_\mathbf{x})). \quad (26)$$

Applying this to the organizing structure function ψ of a modular decomposition (ψ, χ) of (ϕ, C) , assuming only that the processes $\chi_k(\mathbf{X}_{A_k}(\cdot)), k =$

$1, \dots, r$ are independent, we have in particular that $B_\psi^{*j}(\cdot)$ is pointwise greater than or equal to $l_\psi^{**j'}(\cdot)$ for every $j' \geq j$, i.e. for each possible \mathbf{p}_χ we have

$$B_\psi^{*j}(\mathbf{p}_\chi) \geq l_\psi^{**j'}(\mathbf{p}_\chi) \text{ for all } j' \geq j.$$

Now assume also that the processes $X_i(\cdot), i \in A_k$ are independent for each k . We then have that $\mathbf{B}_\chi^*(\mathbf{p}_\mathbf{x}) \geq \mathbf{l}_\chi^{**}(\mathbf{p}_\mathbf{x})$ componentwise. Using that $B_\psi^{*j}(\cdot)$ is non-decreasing in each argument, and inserting the respective lower bounds instead of \mathbf{p}_χ in the above inequality, we obtain

$$p_\phi^j \geq B_\psi^*(\mathbf{B}_\chi^*(\mathbf{p}_\mathbf{x})) \geq l_\psi^{**j'}(\mathbf{l}_\chi^{**}(\mathbf{p}_\mathbf{x})) \text{ for all } j' \geq j. \quad (27)$$

By the same argument we have

$$B_\psi^*(\mathbf{B}_\chi^*(\mathbf{p}_\mathbf{x})) \geq l_\psi^{*j'}(\mathbf{l}'_\chi(\mathbf{p}_\mathbf{x})) \text{ for all } j' \geq j. \quad (28)$$

The argument leading to (27) and (28) can be generalized to deal with other comparisons, based on other distributional assumptions, e.g. the case when the exact availabilities \mathbf{p}_χ of the modules are known, or when the bounds (4) and (5) are available for the modules and the joint performance process of each module is associated. Formally, we have the following theorem:

Theorem 6 *For any $n = 1, 2, \dots$, let $\mathcal{P}^n, \tilde{\mathcal{P}}^n$ be subsets of the sets of distributions $F_\mathbf{X}$ for the joint performance process \mathbf{X} of a system (C, ϕ) with n components. Suppose that*

- (i) *if $F_\mathbf{X} \in \mathcal{P}^n$, then $\lambda_{\phi,1}^j(\cdot), \lambda_{\phi,2}^j(\cdot)$ are lower availability bounds in I for ϕ to level j such that*

$$\lambda_{\phi,1}^j(\mathbf{p}_\mathbf{x}) \leq \lambda_{\phi,2}^j(\mathbf{p}_\mathbf{x}) \text{ for all possible } \mathbf{p}_\mathbf{x}, \text{ and that both } \lambda_{\phi,1}^j \text{ and } \lambda_{\phi,2}^j \text{ are non-decreasing in each argument, and}$$

- (ii) *if $F_\mathbf{X} \in \tilde{\mathcal{P}}^n$, then for all $j = 1, \dots, M$ $\tilde{\lambda}_{\phi,1}^j, \tilde{\lambda}_{\phi,2}^j$ are lower availability bounds in I for ϕ to level j such that*

$$\tilde{\lambda}_{\phi,1}^j \leq \tilde{\lambda}_{\phi,2}^j.$$

Assume that ϕ has the modular decomposition ψ, χ , with n_k components in A_k , and suppose that the distribution of the joint performance process of $(\chi_1(\mathbf{X}_{A_1}), \dots, \chi_r(\mathbf{X}_{A_r}))$ belongs to \mathcal{P}^r , while the distributions of the joint performance processes \mathbf{X}_{A_k} belong to $\tilde{\mathcal{P}}^{n_k}$ for each k . Then

$$\lambda_{\psi,1}^j(\tilde{\lambda}_{\chi,1}) \leq \lambda_{\psi,2}^j(\tilde{\lambda}_{\chi,2})$$

A corresponding result is valid for unavailabilities.

A large number of explicit comparisons based on the general argument behind this theorem are given in Funnemark and Natvig (1985). The typical application is that \mathcal{P}^r denotes the set of distributions for which the r component processes are independent in I , whereas $\tilde{\mathcal{P}}^{n_k}$ denotes either the set of distributions such that the components in \mathbf{X}_{A_k} are independent, the set of distributions such that these components are associated in I , or the set of all possible distributions.

We now return to the inequalities (27) and (28), assuming again independence of the component processes. Combining these inequalities with respectively theorem 5 and theorem 2, and then using the definition (26), we obtain the first inequality of the following corollary, the second inequality being obvious by monotonicity of $B_\psi^*(\cdot)$ in each argument:

Corollary 1 *Let I be an arbitrary interval, and assume that the component processes are independent in I . If the modular decomposition (ψ, χ) of (C, ϕ) has the property described in (iv) of theorem 1, and in particular if the modular structure functions $\chi_k, k = 1, \dots, r$, satisfy the condition (iii) of that theorem, then*

$$B_\phi^*(\mathbf{p}\mathbf{x}) \leq B_\psi^*(\mathbf{B}_\chi^*(\mathbf{p}\mathbf{x})) \leq p_\phi^j$$

This result was given in Funnemark and Natvig (1985) for the case of $I = [t, t]$, but without realising the need for assumptions like those given in theorem 1.

7 Does refinement or coarsening of modular decompositions improve availability bounds?

Corollary 1 tells us that the best bound B^* based on independent component processes is improved by a modular decomposition. The question arises which decomposition is preferable when several modular decompositions are possible. We conclude the paper by a discussion which is motivated by this question, which, however, can not be answered in general. We start by establishing some necessary notation. Let (θ, ω) be a modular decomposition

which is a refinement of (ψ, χ) . By this we mean that $\omega = (\omega_1, \dots, \omega_s)$ is of the form $\omega = (\omega_1, \dots, \omega_r)$, and there exist organizing structure functions $\sigma_1, \dots, \sigma_r$ such that (σ_k, ω_k) is a modular decomposition of χ_k , i.e. $\chi_k(\mathbf{x}_{A_k}) = \sigma_k(\omega_k(\mathbf{x}_{A_k}))$, $k = 1, \dots, r$. Taking the opposite perspective, we say that (ψ, χ) is a coarsening of (θ, ω) . We may then regard θ as the structure function of a system (D, θ) whose components D are the s components of ω . The modular decomposition (ψ, χ) of (C, ϕ) induces the modular decomposition (ψ, σ) of (D, θ) , where by definition $\sigma = (\sigma_1, \dots, \sigma_r)$.

Throughout this section we will assume that the modular structure functions $\chi_k, k = 1, \dots, r$, and $\omega_l, l = 1, \dots, s$ satisfy the operational condition (iii) of theorem 1. Then the modular decompositions (ψ, χ) and (θ, ω) both have property (iv) of that theorem, and the bounds $B_\psi^*(\mathbf{B}_\chi^*(\mathbf{p}\mathbf{x}))$ and $B_\theta^*(\mathbf{B}_\omega^*(\mathbf{p}\mathbf{x}))$ both improve B_ϕ^* according to corollary 1. However, neither of these two bounds is always the better one. But we will show that under these conditions both these bounds can be improved by combining the two decompositions to form

$$B_\psi^*(\mathbf{B}_\sigma^*(\mathbf{B}_\omega^*(\mathbf{p}\mathbf{x}))).$$

In order to prove this, we have to apply corollary 1 to the modular decomposition (ψ, σ) of (D, θ) . Our assumptions on the modular structure functions $\chi_k, k = 1, \dots, r$, ensure that we can do this by the following lemma. The lemma provides an alternative to making assumptions as in theorem 1 directly for this modular decomposition.

Lemma 3 *Suppose that condition (iii) of theorem 1 is satisfied for the modular structure functions $\chi_k, k = 1, \dots, r$, in the modular decomposition (ψ, χ) of (C, ϕ) . Let (θ, ω) be a modular decomposition which is a refinement of (ψ, χ) , with (σ_k, ω_k) a corresponding modular decomposition of $\chi_k, k = 1, \dots, r$. Assume that the components of the process $\omega(\mathbf{X}(\cdot))$ are independent in an interval I . Then*

$$B_\theta^*(\mathbf{p}\omega) \leq B_\psi^*(\mathbf{B}_\sigma^*(\mathbf{p}\omega)).$$

Proof: Considering the system (D, θ) instead of the system (C, ϕ) , the lemma follows from corollary 1, if we can prove that the modular structure functions σ_k also satisfy condition (iii) of theorem 1. Note that σ_k and χ_k take the same values, i.e. $S_{\sigma_k} = S_{\chi_k}$. Suppose $\mathbf{w}_k = \omega_k(\mathbf{x}_{A_k})$ is an mcv for σ_k to level v_k^+ for some $v_k \in S_{\sigma_k}$. Then clearly \mathbf{x}_{A_k} is a cut vector for χ_k to level v_k^+ .

Choose an mcv $\mathbf{z}_{A_k} \geq \mathbf{x}_{A_k}$ to level v_k^+ for χ_k . Then $\boldsymbol{\omega}_k(\mathbf{z}_{A_k})$ is a cut vector for σ_k to level v_k^+ satisfying $\boldsymbol{\omega}_k(\mathbf{z}_{A_k}) \geq \boldsymbol{\omega}_k(\mathbf{x}_{A_k}) = \mathbf{w}_k$. Since \mathbf{w}_k is an mcv for σ_k to the level v_k^+ , it follows that $\boldsymbol{\omega}_k(\mathbf{z}_{A_k}) = \mathbf{w}_k$. Then, by assumption,

$$v_k = \chi_k(\mathbf{z}_{A_k}) = \sigma_k(\boldsymbol{\omega}_k(\mathbf{z}_{A_k})) = \sigma_k(\mathbf{w}_k),$$

as required.

Corollary 2 *Let I be an arbitrary interval, and assume that the component processes are independent in I . Let $(\theta, \boldsymbol{\omega})$ be a refinement of the modular decomposition $(\psi, \boldsymbol{\chi})$ of (C, ϕ) , with $(\sigma_k, \boldsymbol{\omega}_k)$ a corresponding modular decomposition of $\chi_k, k = 1, \dots, r$. Suppose that condition (iii) of theorem 1 is satisfied for the modular structure functions $\chi_k, k = 1, \dots, r$, and $\omega_l, l = 1, \dots, s$. Then*

$$\max(B_\psi^*(\mathbf{B}_\chi^*(\mathbf{p}_\mathbf{x})), B_\theta^*(\mathbf{B}_\omega^*(\mathbf{p}_\mathbf{x}))) \leq B_\psi^*(\mathbf{B}_\sigma^*(\mathbf{B}_\omega^*(\mathbf{p}_\mathbf{x}))).$$

Proof: Consider the first argument in the max-function. By assumption, all the modular structure functions $\omega_l, l = 1, \dots, s$, in the modular decompositions $(\sigma_k, \boldsymbol{\omega}_k)$ of $\chi_k, k = 1, \dots, r$, satisfy condition (iii) of theorem 1. By corollary 1 it follows that $\mathbf{B}_\chi^*(\mathbf{p}_\mathbf{x}) \leq \mathbf{B}_\sigma^*(\mathbf{B}_\omega^*(\mathbf{p}_\mathbf{x}))$ componentwise. By monotonicity of $B_\psi^*(\cdot)$, the first argument in the max-function is taken care of. By replacing \mathbf{p}_ω by $\mathbf{B}_\omega^*(\mathbf{p}_\mathbf{x})$ in the inequality of lemma 3, the second argument in the max-function is taken care of.

Acknowledgement

I am grateful to professor Bent Natvig for inspiring me to look at the problems treated in this paper, and for his helpful comments and careful reading of the manuscript.

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